

On Certain Hypotheses in Optimal Control Theory and the Relationship of the Maximum Principle with the Dynamic Programming Method

Proposed by L. I. Rozonoer

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Abstract. In this paper we will study three hypotheses proposed by L. I. Rozonoer [1] in optimal control theory in order to derive conditions for the existence of an optimal control under all initial conditions, and the relationships between Pontryagin maximum principle and the dynamic programming method.

1 Introduction

Let us introduce the following optimal control problem considered in [1]:

OCP *To minimize the Lagrange cost functional*

$$\int_{t_0}^T F(x, u, t) dt \quad (1.1)$$

subject to the controlled system

$$\dot{x} = f(x, u, t), \quad (1.2)$$

with $u(t) \in U$ and the initial state condition

$$x(t_0) = x^0, \quad (1.3)$$

where t_0 and T with $t_0 < T$ are prescribed real numbers.

In **OCP**, $U \subseteq \mathbb{R}^m$ is the control domain while the set of admissible controls under consideration is the set of all Lebesgue measurable selection $u(t) \in U$ (see also (2.12) or Remark 4.2); $x^0 = (x_1^0, \dots, x_n^0)^T \in \mathbb{R}^n$ is the initial state, $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ is the state variable, and $f = (f_1, \dots, f_n)^T$ is n-dimensional vector-valued function, where

and throughout this paper the superscript T denotes the transpose of a vector or matrix. Other technical assumptions on f and F will be given in the following sections.

The control Hamiltonian for **OCP** is

$$\mathcal{H}(x, p, u, t) := \sum_{i=1}^n p_i f_i(x, u, t) - F(x, u, t), \quad (1.4)$$

where $p = (p_1, \dots, p_n)^T \in \mathbb{R}^n$ is the costate variable.

For any given initial data (x^0, τ) with $\tau \in [t_0, T]$ and $x^0 \in \mathbb{R}^n$, we introduce the control Hamiltonian system

$$\begin{cases} \dot{x} = \frac{\partial \mathcal{H}(x, p, u, t)}{\partial p}, \\ \dot{p} = -\frac{\partial \mathcal{H}(x, p, u, t)}{\partial x}, \end{cases} \quad (1.5)$$

with the two-point boundary value conditions

$$x(\tau) = x^0, \quad p(T) = 0. \quad (1.6)$$

Remark 1.1. In order to distinguish the function (1.4) and the system (1.5) with the Hamiltonian (4.10) and the canonical Hamiltonian system (4.21), which will be considered in Section 4 and very related to these analogues, we prefer to calling (1.4) (and (1.5)) the control Hamiltonian (and the control Hamiltonian system) instead of the Hamiltonian (and the Hamiltonian system).

Definition 1.1. A control $u^*(\cdot) : [\tau, T] \mapsto U$ is said to satisfy the Pontryagin maximum condition on the interval $[\tau, T]$ under the initial data (x^0, τ) , provided that the unique solution $(x^*(\cdot), p^*(\cdot))$ of the control Hamiltonian system (1.5)-(1.6) corresponding to this control $u^*(\cdot)$ satisfy

$$\mathcal{H}(x^*(t), p^*(t), u^*(t), t) \geq \mathcal{H}(x^*(t), p^*(t), u, t), \quad \forall u \in U, \quad \text{a.e. } t \in [\tau, T]. \quad (1.7)$$

Related to **OCP**, the Hamilton-Jacobi-Bellman equation (or called the Bellman equation in [1]) is

$$-v_\tau + \sup_{u \in U} \mathcal{H}(x, -v_x, u, \tau) = 0, \quad (x, \tau) \in \mathbb{R}^n \times (t_0, T), \quad (1.8)$$

with the boundary condition

$$v(x, T) = 0, \quad (1.9)$$

where and throughout this paper, the partial derivative of a given function φ with respect to $\tau \in [t_0, T]$ or $x \in \mathbb{R}^n$ will be denoted by φ_τ or φ_x , respectively.

In order to adapt for the optimal control theory, L. I. Rozonoer [1] first give the following concept of weak solution to the Hamilton-Jacobi-Bellman equation:

Definition 1.2. A continuous function $V : \mathbb{R}^n \times [t_0, T] \mapsto \mathbb{R}$ is said to be an extended solution of the Hamilton-Jacobi-Bellman equation (1.8)-(1.9) provided that, for any $x \in \mathbb{R}^n$ and $\tau \in [t_0, T]$ there exists a p^* with $-p^* \in \partial_+ V_x(x, \tau)$ and a $u^* \in U$ such that

$$\mathcal{H}(x, p^*, u^*, \tau) \geq \mathcal{H}(x, p^*, u, \tau), \quad \forall u \in U, \quad (1.10)$$

and

$$\mathcal{H}(x, p^*, u^*, \tau) \in \partial_+ V_\tau(x, \tau), \quad (1.11)$$

along with the boundary condition (1.9).

The notation $\partial_+ V_x(x, \tau)$ and $\partial_+ V_\tau(x, \tau)$ in **Definition 1.2** means the superdifferential of the function $V(\cdot, \cdot)$ at the point (x, τ) with respect to x and τ , respectively. We will recall the concepts of superdifferential and subdifferential in Section 2.

In order to derive conditions for the existence of an optimal control under all initial conditions, and thereby the relationships between Pontryagin maximum principle and the dynamic programming method, L. I. Rozonoer [1] proposed three hypotheses on **OCP**.

Hypothesis 1. *The existence of an extended solution to the Hamilton-Jacobi-Bellman equation is necessary and sufficient for the existence of an optimal control under all initial data (x^0, τ) .*

Hypothesis 2. *The extended solution of the Hamilton-Jacobi-Bellman equation exists if and only if for every initial data (x^0, τ) , there exists a unique control satisfying the Pontryagin maximum condition.*

Hypothesis 3. *If for every initial data (x^0, τ) , there exists a unique control satisfying the Pontryagin maximum condition, then this control is optimal.*

Just as emphasized in [1, 2] that the concept of solution to the Hamilton-Jacobi-Bellman equation need to be generalized to ensure that certain general hypotheses could be given on the condition for the existence of an optimal control under all initial data, and as a result the relationships between Pontryagin maximum principle and the dynamic programming method. In this approach, [3, 4] generalized the concept of solution of the Hamilton-Jacobi-Bellman equation to help demonstrating the necessary and sufficient conditions for minimization of a functional not only for nonsmooth cases, but also for the case where there is even no optimal control, and provide a possibility for investigating control design. On the other hand, many works such as [5, 6] and the references cited within devoted to the relationships between Pontryagin maximum principle and the dynamic programming method directly or in the framework of viscosity solution theory of the Hamilton-Jacobi-Bellman equation. There are rich references related to this approach (see [1, 6]).

In this paper, we will only focus on the problem **OCP** in order to study these above hypotheses.

The rest of paper is organized as follows: In Section 2, we will study **Hypothesis 1**. First, it will be considered the concept of the extended solution defined by L. I. Rozonoer. Under some mild technical assumptions, the Bellman function is just right an extended solution to the Hamilton-Jacobi-Bellmen equation. Second, one example will be given to show that the Bellman function is not an extended solution but a viscosity solution to the Hamilton-Jacobi-Bellmen equation, which indicates the application range of the extended solution in some sense. Finally, one counterexample will be given to verify that **OCP** may have no optimal controls under some initial data (x^0, τ) even that the Bellman function is a classical (C^2 smooth) solution to the Hamilton-Jacobi-Bellmen equation, which is verified to be an extended solution as well. In Section 3, we will study **Hypothesis 2**. Two counterexamples will be given to show that there are many optimal controls under every initial data (x^0, τ) even that the Bellman function is a classical (C^2 smooth) solution to the Hamilton-Jacobi-Bellmen equation, which is verified to be an extended solution as well. On the other hand, all optimal controls satisfy the Pontryagin maximum condition. In section 4, we will study **Hypothesis 3**. First, it will be given the necessary and sufficient condition for the differentiability of the Hamiltonian. Then, main results will be established that **Hypothesis 3** holds true under some technical assumptions of regularity on the data, through the existing relationships between the Hamiltonian system and the Hamilton-Jacobi equation. In Section 5, the conclusions will be given.

2 On the extended solution and Hypotheses 1

2.1 On the concept of the extended solution

First, we recall some related concepts and results in the theory of the Hamilton-Jacobi-Bellman equations.

Let n be a positive integer. We denote by the operation $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the inner product and norm in \mathbb{R}^n .

The following definition is combined from [7], [8] and [6], etc.

Definition 2.1. For a continuous function $\varphi : \mathbb{R}^n \times [t_0, T] \mapsto \mathbb{R}$, define that

$$\begin{aligned} \partial_+ \varphi_{x,\tau}(x, \tau) &:= \{(p, q) \in \mathbb{R}^{n+1} \mid \limsup_{(y,t) \rightarrow (x,\tau)} \frac{\varphi(y,t) - \varphi(x,\tau) - \langle p, y-x \rangle - q(t-\tau)}{|t-\tau| + \|y-x\|} \leq 0\}, \\ \partial_- \varphi_{x,\tau}(x, \tau) &:= \{(p, q) \in \mathbb{R}^{n+1} \mid \liminf_{(y,t) \rightarrow (x,\tau)} \frac{\varphi(y,t) - \varphi(x,\tau) - \langle p, y-x \rangle - q(t-\tau)}{|t-\tau| + \|y-x\|} \geq 0\}, \\ \partial_+ \varphi_x(x, \tau) &:= \{p \in \mathbb{R}^n \mid \limsup_{y \rightarrow x} \frac{\varphi(y,\tau) - \varphi(x,\tau) - \langle p, y-x \rangle}{\|y-x\|} \leq 0\}, \\ \partial_- \varphi_x(x, \tau) &:= \{p \in \mathbb{R}^n \mid \liminf_{y \rightarrow x} \frac{\varphi(y,\tau) - \varphi(x,\tau) - \langle p, y-x \rangle}{\|y-x\|} \geq 0\}, \\ \partial_+ \varphi_\tau(x, \tau) &:= \{q \in \mathbb{R} \mid \limsup_{t \rightarrow \tau} \frac{\varphi(x,t) - \varphi(x,\tau) - q(t-\tau)}{|t-\tau|} \leq 0\}, \end{aligned} \tag{2.1}$$

for a given $(x, \tau) \in \mathbb{R}^n \times (t_0, T)$.

$\partial_+ \varphi_{x,\tau}(x, \tau)$ and $\partial_- \varphi_{x,\tau}(x, \tau)$ are called the superdifferential and subdifferential of φ at (x, τ) , respectively; $\partial_+ \varphi_x(x, \tau)$ and $\partial_- \varphi_x(x, \tau)$ are called the partial superdifferential and subdifferential of φ at (x, τ) with respect to x , respectively; $\partial_+ \varphi_\tau(x, \tau)$ is called the partial superdifferential of φ at (x, τ) with respect to τ .

Remark 2.1. We can define the right superdifferential $\partial_+\varphi_{x,\tau+}(x, \tau)$, right subdifferential $\partial_-\varphi_{x,\tau+}(x, \tau)$ and partial right superdifferential $\partial_+\varphi_{\tau+}(x, \tau)$ with respect to τ at $(x, \tau) \in \mathbb{R}^n \times [t_0, T)$ by restricting $t \downarrow \tau$ in (2.1). Analogously, define the left superdifferential $\partial_+\varphi_{x,\tau-}(x, \tau)$, left subdifferential $\partial_-\varphi_{x,\tau-}(x, \tau)$ and partial left superdifferential $\partial_+\varphi_{\tau-}(x, \tau)$ at $(x, \tau) \in \mathbb{R}^n \times (t_0, T]$ by restricting $t \uparrow \tau$ in (2.1).

From the above definition, it can be easily deduced that

Lemma 2.1. *Let $\varphi : \mathbb{R}^n \times [t_0, T] \mapsto \mathbb{R}$ be continuous. It holds that*

- (a) *if $(p, q) \in \partial_+\varphi_{x,\tau}(x, \tau)$, then $p \in \partial_+\varphi_x(x, \tau)$ and $q \in \partial_+\varphi_\tau(x, \tau)$;*
- (b) *if $(p, q) \in \partial_+\varphi_{x,\tau+}(x, t_0)$, then $p \in \partial_+\varphi_x(x, t_0)$ and $q \in \partial_+\varphi_{\tau+}(x, t_0)$;*
- (c) *if $(p, q) \in \partial_+\varphi_{x,\tau-}(x, T)$, then $p \in \partial_+\varphi_x(x, T)$ and $q \in \partial_+\varphi_{\tau-}(x, T)$.*

The definition of the viscosity solution to the first order PDEs is first given by Crandall and Lions [9]. We can also refer to [7], [8], [10] and [6], etc., for the following definition.

Definition 2.2. *A continuous function $\varphi : \mathbb{R}^n \times [t_0, T] \mapsto \mathbb{R}$ is called a viscosity subsolution of the Hamilton-Jacobi-Bellman equation (1.8) provided that, for any $(x, \tau) \in \mathbb{R}^n \times (t_0, T)$,*

$$-q + \sup_{u \in U} \mathcal{H}(x, -p, u, \tau) \leq 0, \quad \forall (p, q) \in \partial_+\varphi_{x,\tau}(x, \tau); \quad (2.2)$$

A continuous function $\varphi : \mathbb{R}^n \times [t_0, T] \mapsto \mathbb{R}$ is called a viscosity supersolution of the Hamilton-Jacobi-Bellman equation (1.8) provided that, for any $(x, \tau) \in \mathbb{R}^n \times (t_0, T)$,

$$-q + \sup_{u \in U} \mathcal{H}(x, -p, u, \tau) \geq 0, \quad \forall (p, q) \in \partial_-\varphi_{x,\tau}(x, \tau). \quad (2.3)$$

Finally, φ is called a viscosity solution of the Hamilton-Jacobi-Bellman equation (1.8) if it is simultaneously a viscosity sub- and supersolution. In addition, if φ satisfies the boundary condition (1.9), then φ is called a viscosity solution of the Hamilton-Jacobi-Bellman equation (1.8)-(1.9).

Denote $\mathbb{R}_+ = [0, +\infty)$. For any $R > 0$, we denote by $B_R(\mathbb{R}^{n+1})$ (or $B_R(\mathbb{R}^n)$) the open ball in \mathbb{R}^{n+1} (or \mathbb{R}^n) with a radius R centered at 0.

Definition 2.3. *Consider a convex subset $K \subseteq \mathbb{R}^{n+1}$ (or \mathbb{R}^n). A function $\varphi : K \mapsto \mathbb{R}$ is called semiconcave if there exists a function $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfies that*

$$\begin{cases} \omega(r, d) \leq \omega(R, D), & \forall r \leq R, d \leq D, \\ \lim_{D \rightarrow 0+} \omega(R, D) = 0, & \forall R > 0, \end{cases} \quad (2.4)$$

such that, for every $R > 0$, $\lambda \in [0, 1]$ and any $\xi, \eta \in K \cap B_R(\mathbb{R}^{n+1})$ (or $K \cap B_R(\mathbb{R}^n)$),

$$\lambda\varphi(\xi) + (1 - \lambda)\varphi(\eta) - \varphi[\lambda\xi + (1 - \lambda)\eta] \leq \lambda(1 - \lambda)\|\xi - \eta\|\omega(R, \|\xi - \eta\|). \quad (2.5)$$

We call the above function ω a modulus of a semiconcavity of φ .

Remark 2.2. Obviously, if $\varphi : \mathbb{R}^n \times [t_0, T] \mapsto \mathbb{R}$ is semiconcave, then both $\varphi(\cdot, t_0) : \mathbb{R}^n \mapsto \mathbb{R}$ and $\varphi(\cdot, T) : \mathbb{R}^n \mapsto \mathbb{R}$ are semiconcave.

By Rademacher's theorem ([11], Ch.5, p.281), if $\varphi : \mathbb{R}^n \times [t_0, T] \mapsto \mathbb{R}$ is locally Lipschitz continuous, then φ is differentiable almost everywhere in $\mathbb{R}^n \times [t_0, T]$. Meanwhile,

$$D^*\varphi(x, \tau) := \left\{ \lim_{i \rightarrow +\infty} \varphi_{x, \tau}(x_i, \tau_i) \mid \mathcal{D}(\varphi) \ni (x_i, \tau_i) \rightarrow (x, \tau) \right. \\ \left. \text{such that } \lim_{i \rightarrow +\infty} \varphi_{x, \tau}(x_i, \tau_i) \text{ exists} \right\},$$

is nonempty at all $(x, \tau) \in \mathbb{R}^n \times [t_0, T]$, where

$$\mathcal{D}(\varphi) := \{(x, \tau) \in \mathbb{R}^n \times (t_0, T) \mid \varphi \text{ is differentiable at } (x, \tau)\}.$$

Lemma 2.2. *Let $\varphi : \mathbb{R}^n \times [t_0, T] \mapsto \mathbb{R}$ be locally Lipschitz continuous. If φ is semiconcave, then it holds that*

(a) *for all $(x, \tau) \in \mathbb{R}^n \times (t_0, T)$,*

$$\partial_+ \varphi_{x, \tau}(x, \tau) = \text{co} D^* \varphi(x, \tau) \neq \emptyset,$$

where the operation “co” denotes the convex hull;

(b)

$$\partial_+ \varphi_{x, \tau+}(x, t_0) \supseteq \text{co} D^* \varphi(x, t_0) \neq \emptyset,$$

and

$$\partial_+ \varphi_{x, \tau-}(x, T) \supseteq \text{co} D^* \varphi(x, T) \neq \emptyset.$$

Proof Part (a) is one part of Theorem 3.3.6 in [8] or in [7]. Part (b) follows easily from the locally Lipschitz continuity and the semiconcavity of φ on $\mathbb{R}^n \times [t_0, T]$, similar to the proofs of Proposition 3.3.1 and 3.3.4 in [8]. \square

Remark 2.3. The semiconcavity of $\varphi : \mathbb{R}^n \times [t_0, T] \mapsto \mathbb{R}$ implies the locally Lipschitz continuity only on $\mathbb{R}^n \times (t_0, T)$. (see [8, 7])

In this section, we will need some technical assumptions on $f : \mathbb{R}^n \times U \times [t_0, T] \mapsto \mathbb{R}^n$ and $F : \mathbb{R}^n \times U \times [t_0, T] \mapsto \mathbb{R}$ as follows:

(H1) Both f and F are continuous, and there exists a constant $M > 0$ such that

$$\|f(0, u, t)\| \leq M, \quad \forall (u, t) \in U \times [t_0, T], \quad (2.6)$$

and

$$|F(0, u, t)| \leq M, \quad \forall (u, t) \in U \times [t_0, T]. \quad (2.7)$$

(H2) Both $f(\cdot, u, \cdot)$ and $F(\cdot, u, \cdot)$ are locally Lipschitz continuous on $\mathbb{R}^n \times [t_0, T]$ uniformly in $u \in U$, i.e., there exists a nondecreasing function $L : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that, for any $(x, t), (y, s) \in B_R(\mathbb{R}^n) \times [t_0, T]$, and any $u \in U$,

$$\|f(x, u, t) - f(y, u, s)\| \leq L(R)(|t - s| + \|x - y\|), \quad (2.8)$$

and

$$|F(x, u, t) - F(y, u, s)| \leq L(R)(|t - s| + \|x - y\|). \quad (2.9)$$

(H3) There exists a modulus $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying (2.4) such that, for any $\lambda \in [0, 1]$ and $(x, t), (y, s) \in B_R(\mathbb{R}^n) \times [t_0, T]$, and any $u \in U$,

$$\begin{aligned} & \|\lambda f(x, u, t) + (1 - \lambda)f(y, u, s) - f[\lambda x + (1 - \lambda)y, u, \lambda t + (1 - \lambda)s]\| \\ & \leq \lambda(1 - \lambda)(|t - s| + \|x - y\|)\omega(R, |t - s| + \|x - y\|). \end{aligned} \quad (2.10)$$

(H4) $F(\cdot, u, \cdot)$ is semiconcave on $\mathbb{R}^n \times [t_0, T]$, i.e., there exists a modulus $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying (2.4) such that, for any $\lambda \in [0, 1]$ and $(x, t), (y, s) \in B_R(\mathbb{R}^n) \times [t_0, T]$, and any $u \in U$,

$$\begin{aligned} & \lambda F(x, u, t) + (1 - \lambda)F(y, u, s) - F[\lambda x + (1 - \lambda)y, u, \lambda t + (1 - \lambda)s] \\ & \leq \lambda(1 - \lambda)(|t - s| + \|x - y\|)\omega(R, |t - s| + \|x - y\|). \end{aligned} \quad (2.11)$$

Remark 2.4. **(H3)** holds true in particular when f is continuously differentiable with respect to (x, t) uniformly in u . More precisely, if we assume that there exists a modulus $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying (2.4) such that, for any $(x, t), (y, s) \in B_R(\mathbb{R}^n) \times [t_0, T]$, and any $u \in U$,

$$\|f_t(x, u, t) - f_t(y, u, s)\| + \|f_x(x, u, t) - f_x(y, u, s)\| \leq \omega(R, |t - s| + \|x - y\|).$$

Conversely, under the assumption **(H2)**, it follows from Proposition 1.1.13 in [10] that, **(H3)** implies that f is continuously differentiable with respect to (x, t) .

Throughout this paper, we define the set of all admissible controls under any given initial time $\tau \in [t_0, T]$, as follows:

$$\mathcal{U}(\tau) := \{u(\cdot) : [\tau, T] \mapsto U \mid u(\cdot) \text{ is Lebesgue measurable}\}, \quad (2.12)$$

and denote by \mathcal{U} simply for $\mathcal{U}(t_0)$. For any given initial data $(x^0, \tau) \in \mathbb{R}^n \times [t_0, T]$, $x(\cdot; x^0, \tau, u(\cdot))$ is the unique solution of the control system (1.2) under the initial state condition $x(\tau) = x^0$ and the control $u(\cdot) \in \mathcal{U}(\tau)$, which we will only denote by $x(\cdot)$ for short if without confusion. The value function (or called the Bellman function in [1]) $V(\cdot, \cdot) : \mathbb{R}^n \times [t_0, T] \mapsto \mathbb{R}$ is defined by

$$V(x^0, \tau) := \inf_{u(\cdot) \in \mathcal{U}(\tau)} \int_{\tau}^T F(x(t), u(t), t) dt, \quad (2.13)$$

where $x(\cdot)$ is the solution of (1.2) under the initial state condition $x(\tau) = x^0$ and the control $u(\cdot) \in \mathcal{U}(\tau)$.

Similar to Theorem 4.1 in [12] (or [10, 8] etc.), it follows that

Proposition 2.1. *Assume that (H1)-(H2) are satisfied. Then the value function $V(\cdot, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}^n \times [t_0, T]$.*

Similar to Theorem 3.2 in [12] (or [10, 8] etc.), it follows that

Proposition 2.2. *Assume that (H1)-(H4) are satisfied. Then the value function $V(\cdot, \cdot)$ is semiconcave on $\mathbb{R}^n \times [t_0, T]$.*

Remark 2.5. The technical assumptions (H1)-(H2) in Proposition 2.1 and (H1)-(H4) in Proposition 2.2 can be weaken in some approaches, for example, the consideration of the cases with unbounded control variables in [12], etc.

Theorem 2.1. *Assume that (H1)-(H4) are satisfied. If the control domain U is compact, then the value function $V(\cdot, \cdot) : \mathbb{R}^n \times [t_0, T] \mapsto \mathbb{R}$ is an extended solution of the Hamilton-Jacobi-Bellman equation (1.8)-(1.9).*

Proof It is well known that, V is the unique viscosity solution to the Hamilton-Jacobi-Bellman equation (1.8)-(1.9).

Proposition 2.1 and 2.2 yields that the value function V is locally Lipschitz continuous and semiconcave on $\mathbb{R}^n \times [t_0, T]$.

By Rademacher's theorem ([11], Ch.5, p.281), the locally Lipschitz continuity of the value function V implies that V is differentiable almost everywhere in $\mathbb{R}^n \times (t_0, T)$. Hence, for any given $(x, \tau) \in \mathbb{R}^n \times [t_0, T]$, there exists a sequence of $\{(x_i, \tau_i)\} \subset \mathbb{R}^n \times (t_0, T)$ such that

$$(x_i, \tau_i) \rightarrow (x, \tau), \quad \text{as } i \rightarrow +\infty,$$

and V is differentiable at all (x_i, τ_i) . According to Proposition 1.9 in [7] (or Theorem 1 in [11], Ch.10, p.545), the value function V satisfies the Hamilton-Jacobi-Bellman equation (1.8) at (x_i, τ_i) in the classical sense, i.e.,

$$-V_\tau(x_i, \tau_i) + \sup_{u \in U} \mathcal{H}(x_i, -V_x(x_i, \tau_i), u, \tau_i) = 0. \quad (2.14)$$

Meanwhile, due to the compactness of U , there exists a sequence of $\{u_i\} \subset U$ such that

$$-V_\tau(x_i, \tau_i) + \mathcal{H}(x_i, -V_x(x_i, \tau_i), u_i, \tau_i) = 0, \quad (2.15)$$

and there exists a subsequence of $\{u_i\}$ (still denoted by themselves without loss of generality) such that

$$u_i \rightarrow u^* \in U, \quad \text{as } i \rightarrow +\infty.$$

Denote that

$$p_i := -V_x(x_i, \tau_i), \quad q_i := V_\tau(x_i, \tau_i).$$

It follows from Lemma 2.2 that, there exists a subsequence of $\{(-p_i, q_i)\}$ (still denoted by themselves without loss of generality) such that

$$V_{x,\tau}(x_i, \tau_i) \equiv (-p_i, q_i) \rightarrow (-p^*, q^*) \in \begin{cases} \partial_+ V_{x,\tau}(x, \tau), & \text{if } \tau \in (t_0, T), \\ \partial_+ V_{x,\tau+}(x, t_0), & \text{if } \tau = t_0, \\ \partial_+ V_{x,\tau-}(x, T), & \text{if } \tau = T, \end{cases} \quad (2.16)$$

as $i \rightarrow +\infty$, where $V_{x,\tau}(x_i, \tau_i)$ is the derivative of $V(\cdot, \cdot)$ with respect to (x, τ) at (x_i, τ_i) .

By Lemma 2.1, combining (2.15) and (2.16) yields the conclusions. \square

2.2 An example for the concept of the extended solution

Example. Let the control domain be $U = [0, 1]$. For any given initial data (x^0, τ) with $\tau \in [0, 1)$ and $x^0 \in \mathbb{R}$. Consider the following linear one-dimensional control system

$$\frac{dx}{dt} = x(t) + u(t), \quad t \in [\tau, 1], \quad (2.17)$$

with the initial condition

$$x(\tau) = x^0 \in \mathbb{R}, \quad (2.18)$$

and let the associated cost functional be

$$J(x^0, \tau; u(\cdot)) = \int_{\tau}^1 |x(t)| dt, \quad (2.19)$$

where the set of all admissible controls is

$$\mathcal{U}(\tau) := \{u(\cdot) : [\tau, 1] \mapsto U \mid u(\cdot) \text{ is Lebesgue measurable}\}. \quad (2.20)$$

Obviously, this optimal control problem has a unique optimal control

$$u^*(t) \equiv \begin{cases} 0, & \text{if } x^0 \geq 0, \\ 1, & \text{if } x^0 < 0, (1 + x^0)e^{1-\tau} \leq 1, \end{cases} \quad (2.21)$$

while

$$u^*(t) \equiv \begin{cases} 1, & t \in [0, \tau - \ln(1 + x^0)), \\ 0, & t \in [\tau - \ln(1 + x^0), 1], \end{cases} \quad (2.22)$$

if $x^0 < 0$ and $(1 + x^0)e^{1-\tau} > 1$.

The corresponding Hamilton-Jacobi-Bellman equation is

$$\begin{cases} -\frac{\partial v}{\partial \tau} + \sup_{u \in U} \{(x + u)\frac{\partial v}{\partial x} - |x|\} = 0, & (x, \tau) \in \mathbb{R} \times (0, 1), \\ v(x, 1) = 0. \end{cases} \quad (2.23)$$

Obviously, the Hamilton-Jacobi-Bellman equation (2.23) has a unique viscosity solution

$$V(x, \tau) = \begin{cases} x(e^{1-\tau} - 1), & \text{if } x \geq 0, \\ 2 + x - \tau - (1 + x)e^{1-\tau}, & \text{if } x < 0, (1 + x)e^{1-\tau} \leq 1, \\ x - \ln(1 + x), & \text{if } x < 0, (1 + x)e^{1-\tau} > 1, \end{cases} \quad (2.24)$$

which is just the value function.

For any given $\tau_0 \in (0, 1)$, we have

$$V(x, \tau_0) = \begin{cases} x(e^{1-\tau_0} - 1), & \text{if } x \geq 0, \\ x - \ln(1 + x), & \text{if } e^{\tau_0-1} - 1 < x < 0, \end{cases} \quad (2.25)$$

which implies that $\partial_+ V_x(0, \tau_0) = \emptyset$ and $\partial_+ V_{x,\tau}(0, \tau_0) = \emptyset$.

Hence $V(\cdot, \cdot)$ defined by (2.24) is a viscosity solution but not an extended solution of the Hamilton-Jacobi-Bellman equation (2.23).

We notice that $V(\cdot, \cdot)$ is not a semiconcave function according to Lemma 2.2.

2.3 An example for Hypotheses 1

We consider the following example of optimal control problem, which is adapted from [13] (Ch.3, p.246).

Example. Let the control domain be $U = [-1, 1]$, and the set of all control variables be

$$\mathcal{U} := \{u(\cdot) : [0, 1] \mapsto U \mid u(\cdot) \text{ is Lebesgue measurable}\}. \quad (2.26)$$

Consider the one-dimensional control system

$$\frac{dx}{dt} = u(t), \quad t \in [0, 1], \quad (2.27)$$

with the initial state condition

$$x(0) = x^0 \in \mathbb{R}, \quad (2.28)$$

and let the associated Lagrange type cost functional be

$$J(x_0; u(\cdot)) = \int_0^1 [x^2(t) - u^2(t)] dt. \quad (2.29)$$

For this optimal control problem, the corresponding Hamilton-Jacobi-Bellman equation is

$$\begin{cases} -\frac{\partial v}{\partial \tau} + |\frac{\partial v}{\partial x}| - x^2 + 1 = 0, & (x, \tau) \in \mathbb{R} \times (0, 1), \\ v(x, 1) = 0. \end{cases} \quad (2.30)$$

It is easy to verify that, the Hamilton-Jacobi-Bellman equation (2.30) admits a C^2 solution

$$V(x, \tau) = \begin{cases} \frac{1}{3}[x^3 - (x + \tau - 1)^3] + \tau - 1, & \text{if } x + \tau \geq 1, \\ \frac{1}{3}x^3 + \tau - 1, & \text{if } x + \tau < 1, x \geq 0, \\ -\frac{1}{3}x^3 + \tau - 1, & \text{if } -x + \tau < 1, x < 0, \\ \frac{1}{3}[-x^3 - (-x + \tau - 1)^3] + \tau - 1, & \text{if } -x + \tau \leq -1, \end{cases} \quad (2.31)$$

which is just the value function related to this optimal control problem. Certainly, this solution is also an extended solution of the HJB equation since the control Hamiltonian is

$$\mathcal{H}(x, p, u, t) = pu - x^2 + u^2, \quad (2.32)$$

and

$$\max_{u \in [-1, 1]} \mathcal{H}(x, p, u, t) = |p| - x^2 + 1, \quad (2.33)$$

is attainable at $u = -1$ or $u = 1$.

However, it can be proved similarly to [13] (Ch. 3, p.247) that there exists no optimal control under the initial data $(0, \tau)$ with $\tau \in [0, 1)$. In these cases, $V(0, \tau) = -1 + \tau$, which is not attainable. In fact, there exists no optimal control under any initial data (x^0, τ) with $\tau \in [0, 1)$ and $|x^0| + \tau < 1$.

3 On Hypotheses 2

3.1 The first example

Consider the special cases of **OCP** with the integrand in the cost functional (1.1) satisfies that

$$F(x, u, t) \equiv C, \quad \forall (x, u, t) \in \mathbb{R}^n \times U \times [t_0, T], \quad (3.1)$$

for some constant $C \in \mathbb{R}$. Meanwhile, let $f : \mathbb{R}^n \times U \times [t_0, T] \mapsto \mathbb{R}^n$ satisfies the following assumption: Both f and f_x are continuous on $\mathbb{R}^n \times U \times [t_0, T]$, and there exists a $L > 0$ such that

$$\begin{cases} \|f(0, u, t)\| \leq L, \\ \|f_x(x, u, t)\| \leq L, \end{cases} \quad \forall (x, u, t) \in \mathbb{R}^n \times U \times [t_0, T]. \quad (3.2)$$

In these cases, any control $u(\cdot) : [\tau, T] \mapsto U$ is an optimal control under the initial data (x^0, τ) , while the Hamilton-Jacobi-Bellman equation

$$-v_\tau + \sup_{u \in U} \left[-\sum_{i=1}^n f_i(x, u, \tau) v_{x_i} - C \right] = 0, \quad (3.3)$$

with the boundary condition

$$v(x, T) = 0, \quad (3.4)$$

admits a classical solution $V(x, \tau) = C(T - \tau)$, which is obviously an extended solution of (3.3)-(3.4).

On the other hand, according to Pontryagin maximum principle ([14]), any optimal control $u(\cdot)$ satisfies the Pontryagin maximum condition.

Therefore, the Hamilton-Jacobi-Bellman equation (3.3)-(3.4) has an extended solution while more than one controls satisfies the Pontryagin maximum condition.

3.2 The second example

Consider the special case of **OCP** with $U = \mathbb{R}^m$ and the quadratic cost functional, which is defined by

$$F(x, u, t) = u^T S u, \quad (3.5)$$

where $S \in \mathbb{R}^{m \times m}$ is nonnegative semi-definite, i.e., $u^T S u \geq 0$ for any $u \in \mathbb{R}^m$ and there exists $u_0 \neq 0$ such that $u_0^T S u_0 = 0$. Meanwhile, let $f : \mathbb{R}^n \times U \times [t_0, T] \mapsto \mathbb{R}^n$ satisfies the same assumption in the previous example.

In this case, the Hamilton-Jacobi-Bellman equation

$$-\frac{\partial v}{\partial \tau} + \sup_{u \in U} \left[-\sum_{i=1}^n f_i(x, u, \tau) \frac{\partial v}{\partial x_i} - u^T S u \right] = 0, \quad (3.6)$$

with the boundary condition

$$v(x, T) = 0, \quad (3.7)$$

admits a classical solution $V(x, \tau) \equiv 0$, which is obviously an extended solution of (3.6)-(3.7).

For any constant $k \in \mathbb{R}$, the control $u(t) = ku_0$ is an optimal control under any initial data (x_0, τ) . According to Pontryagin maximum principle ([14]), any optimal control $u(\cdot)$ satisfies the Pontryagin maximum condition.

In general, **Hypothesis 2** may not hold for **OCP**.

4 On Hypotheses 3

4.1 Some Preparations

Let $\phi : \mathbb{R}^n \times U \mapsto \mathbb{R}$ be a given function.

$$\Phi(x) := \sup_{u \in U} \{\phi(x, u)\}, \quad \forall x \in \mathbb{R}^n, \quad (4.1)$$

is an extended function, i.e., taking values in $\mathbb{R} \cup \{+\infty\}$. For any $x \in \mathbb{R}^n$,

$$M(x) = \arg \min_{u \in U} \phi(x, u) := \{u \in U \mid \phi(x, u) = \Phi(x)\}, \quad (4.2)$$

which is possibly empty.

Lemma 4.1. *Assume that $\Phi(x)$ is finite for all $x \in \mathbb{R}^n$, and there exists a modulus $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying (2.4) such that,*

$$|\phi(x, u) - \phi(y, u)| \leq \omega(R, \|x - y\|), \quad \forall x, y \in B_R(\mathbb{R}^n), \quad u \in U, \quad (4.3)$$

where $B_R(\mathbb{R}^n)$ denotes the open ball in \mathbb{R}^n with a radius $R > 0$ centered at 0.

Then it holds that

$$\begin{aligned} \partial_- \Phi(x) &\supseteq \partial_- \phi_x(x, u), \\ \partial_+ \phi_x(x, u) &\supseteq \partial_+ \Phi(x), \end{aligned} \quad \text{for all } u \in M(x). \quad (4.4)$$

Proof Since $-\Phi(x) = \inf_{u \in U} [-\phi(x, u)]$, applying Lemma 2.11 in [7] yields the conclusions. \square

Lemma 4.2. *Assume that U is compact, and $\phi : \mathbb{R}^n \times U \mapsto \mathbb{R}$ satisfies (4.3), and*

(A) *$\phi(\cdot, u)$ is differentiable at $x \in \mathbb{R}$ uniformly in $u \in U$, i.e., there exists some modulus $\omega_1 : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with*

$$\lim_{R \rightarrow 0} \omega_1(R) = 0, \quad (4.5)$$

such that

$$|\phi(x + \Delta x, u) - \phi(x, u) - \frac{\partial \phi}{\partial x}(x, u) \Delta x| \leq \|\Delta x\| \omega_1(\|\Delta x\|), \quad (4.6)$$

for small Δx and all $u \in U$;

(B) $\frac{\partial}{\partial x}\phi(x, \cdot) : U \mapsto \mathbb{R}^n$ is continuous;

(C) $\phi(x, \cdot) : U \mapsto \mathbb{R}$ is lower semicontinuous.

Then $M(x) \neq \emptyset$, and

$$\partial_- \Phi(x) = \overline{\text{co}} Y(x), \quad (4.7)$$

where $Y(x) := \{\frac{\partial}{\partial x}\phi(x, u) \mid u \in M(x)\}$;

$$\partial_+ \Phi(x) = \begin{cases} Y(x), & \text{if } Y(x) \text{ is a singleton,} \\ \emptyset, & \text{if } Y(x) \text{ is not a singleton.} \end{cases} \quad (4.8)$$

In particular, Φ is differentiable at x if and only if $Y(x)$ is a singleton.

Moreover, Φ has the directional derivative in any direction $v \in \mathbb{R}^n$, given by

$$\frac{\partial \Phi}{\partial v}(x) = \max_{u \in M(x)} \frac{\partial \phi}{\partial x}(x, u)v = \max_{p \in \partial_- \Phi(x)} \langle p, v \rangle. \quad (4.9)$$

Proof Since $-\Phi(x) = \inf_{u \in U} [-\phi(x, u)]$, applying Proposition 2.13 in [7] yields the conclusions. \square

Definition 4.1. The function $H : \mathbb{R}^n \times \mathbb{R}^n \times [t_0, T] \mapsto \mathbb{R}$ defined by

$$H(x, p, t) := \sup_{u \in U} \mathcal{H}(x, p, u, t) \equiv \sup_{u \in U} \sum_{i=1}^n \{p_i f_i(x, u, t) - F(x, u, t)\}, \quad (4.10)$$

is called the Hamiltonian related to **OCP**, where the functions $f = (f_1, \dots, f_n)^T : \mathbb{R}^n \times U \times [t_0, T] \mapsto \mathbb{R}^n$ and $F : \mathbb{R}^n \times U \times [t_0, T] \mapsto \mathbb{R}$ are the data of **OCP**.

In this section, we will need some technical assumptions on $f : \mathbb{R}^n \times U \times [t_0, T] \mapsto \mathbb{R}^n$ and $F : \mathbb{R}^n \times U \times [t_0, T] \mapsto \mathbb{R}$ as follows:

(H5) Both f and F are continuous, and there exists an absolute modulus $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying (2.4) such that,

$$|\phi(x, u) - \phi(y, u)| \leq \omega(R, \|x - y\|), \quad \forall x, y \in B_R(\mathbb{R}^n), \quad u \in U, \quad (4.11)$$

with $\phi : \mathbb{R}^n \times U \mapsto \mathbb{R}$ being

$$\phi(x, u) = f_1(x, u, t), \text{ or } \dots, \text{ or } f_n(x, u, t), \text{ or } F(x, u, t), \quad (4.12)$$

for any given $t \in [t_0, T]$.

(H6) $\frac{\partial}{\partial x} f : \mathbb{R}^n \times U \times [t_0, T] \mapsto \mathbb{R}^{n \times n}$ and $\frac{\partial}{\partial x} F : \mathbb{R}^n \times U \times [t_0, T] \mapsto \mathbb{R}^n$ are continuous, and there exists an absolute modulus $\omega_1 : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying (4.5) such that $\phi : \mathbb{R}^n \times U \mapsto \mathbb{R}$ in (H5) satisfy (4.6), for any given $t \in [t_0, T]$.

Proposition 4.1. Assume that (H5) holds, and both f and F are differentiable with respect to $x \in \mathbb{R}^n$. Then it holds that

(I) $H(x, \cdot, t)$ is convex;

(II) If

$$A(x, p, t) := \{u \in U \mid \mathcal{H}(x, p, u, t) = H(x, p, t)\} \neq \emptyset, \quad (4.13)$$

and $H(x, \cdot, t)$ is differentiable at $p \in \mathbb{R}$, then

$$f(x, u, t) \equiv c, \quad \text{on} \quad A(x, p, t); \quad (4.14)$$

If $A(x, p, t) \neq \emptyset$ and $H(\cdot, p, t)$ is differentiable at $p \in \mathbb{R}$, then

$$\frac{\partial}{\partial x} f(x, u, t)p - \frac{\partial}{\partial x} F(x, u, t) \equiv c, \quad \text{on} \quad A(x, p, t). \quad (4.15)$$

Proof (I) Let $\mathcal{H}(x, p, u, t) = \langle f(x, u, t), p \rangle - f^0(x, u, t)$, we have

$$\begin{aligned} H(x, \lambda p_1 + (1 - \lambda)p_2, t) &= \sup_{u \in U} \mathcal{H}(x, \lambda p_1 + (1 - \lambda)p_2, u, t) \\ &= \sup_{u \in U} [\lambda \mathcal{H}(x, p_1, u, t) + (1 - \lambda) \mathcal{H}(x, p_2, u, t)] \\ &\leq \lambda H(x, p_1, t) + (1 - \lambda) H(x, p_2, t), \end{aligned}$$

for any $\lambda \in [0, 1]$.

(II) Lemma 4.1 yields the conclusions. \square

Corollary 4.1. Assume that f and F satisfy the assumptions in Proposition 4.1. If $H(\cdot, \cdot, t)$ is differentiable and $A(x, p, t) \neq \emptyset$ at some $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$. Then

$$\begin{cases} \frac{\partial}{\partial p} H(x, p, t) = \{f(x, u, t) \mid \mathcal{H}(x, p, u, t) = H(x, p, t)\} \\ \frac{\partial}{\partial x} H(x, p, t) = \{\frac{\partial}{\partial x} f(x, u, t)p - \frac{\partial}{\partial x} F(x, u, t) \mid \mathcal{H}(x, p, u, t) = H(x, p, t)\}. \end{cases} \quad (4.16)$$

This corollary is just Proposition 3.2 in [15].

Proposition 4.2. Assume that (H5)-(H6) holds, and the control domain U is compact. If

$$A(x, p, t) := \{u \in U \mid \mathcal{H}(x, p, u, t) = H(x, p, t)\} \neq \emptyset, \quad (4.17)$$

and

$$f(x, u, t) \equiv c, \quad \text{on} \quad A(x, p, t), \quad (4.18)$$

then $H(x, \cdot, t)$ is differentiable at $p \in \mathbb{R}$; If $A(x, p, t) \neq \emptyset$ and

$$\frac{\partial}{\partial x} f(x, u, t)p - \frac{\partial}{\partial x} F(x, u, t) \equiv c, \quad \text{on} \quad A(x, p, t), \quad (4.19)$$

then $H(\cdot, p, t)$ is differentiable at $x \in \mathbb{R}$.

Proof Lemma 4.2 yields the conclusions. \square

4.2 Main Results

It is well-known that, under some convex assumptions of the data f and F , the necessary condition – Pontryagin maximum principle is also sufficient (see [6, 16] etc.). According to Theorem 2.5 in [6], we have

Proposition 4.3. *Assume that the control domain U is convex and $\mathcal{H}(\cdot, p, \cdot, t) : \mathbb{R}^n \times U \mapsto \mathbb{R}$ is concave for all $(p, t) \in \mathbb{R}^n \times [t_0, T]$. Then $u^* : [\tau, T] \mapsto U$ is an optimal control of **OCP** under the initial condition $x(\tau) = x^0$ with $(x^0, \tau) \in \mathbb{R}^n \times [t_0, T]$, if and only if u^* satisfies Pontryagin maximum principle, i.e.,*

$$\mathcal{H}(x^*(t), p^*(t), u^*(t), t) = \max_{u \in U} \mathcal{H}(x^*(t), p^*(t), u, t), \quad \text{a.e. } t \in [\tau, T], \quad (4.20)$$

where $(x^*(\cdot), p^*(\cdot))$ is the unique solution to the control Hamiltonian system (1.5)-(1.6).

In particular, **Hypotheses 3** is true.

Remark It is obviously the fact that, **Hypotheses 3** is true provided that there exists an optimal control under all initial conditions. There exist many references on the existence of optimal controls such as the famous Cesari' type conditions [17], etc..

Now we will consider **Hypotheses 3** directly through Hamilton-Jacobi Theory based on some regularity condition of the data f and F instead of the above convex conditions.

Definition 4.2. Let $H : \mathbb{R}^n \times \mathbb{R}^n \times [t_0, T] \mapsto \mathbb{R}$ be defined in Definition 4.1. The system

$$\begin{cases} \dot{x} = \frac{\partial}{\partial p} H(x, p, t), \\ \dot{p} = -\frac{\partial}{\partial x} H(x, p, t), \end{cases} \quad p(T) = 0, \quad (4.21)$$

is called the Hamiltonian system related to **OCP**.

Definition 4.3. The Hamiltonian system (4.21) is called a complete system provided that, for any $\xi \in \mathbb{R}^n$, (4.21) under the terminal condition

$$x(T) = \xi, \quad (4.22)$$

has a unique solution $(x(\cdot), p(\cdot))$ on $[t_0, T]$; Moreover, for any sequence of solutions $(x^i(\cdot), p^i(\cdot))$ of (4.21) with

$$\lim_{i \rightarrow +\infty} (x^i(\tau_i), p^i(\tau_i), \tau_i) = (\xi, \eta, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times [t_0, T],$$

it holds that $(x^i(\cdot), p^i(\cdot))$ converge uniformly on $[t_0, T]$ to the solution $(x(\cdot), p(\cdot))$ of (4.21) with $(x(\tau), p(\tau)) = (\xi, \eta)$.

Definition 4.4. The Hamiltonian system (4.21) related to **OCP** is called to have a shock at time $\tau \in [t_0, T)$ if there exist two solution $(x^i, p^i) : [t_0, T] \mapsto \mathbb{R}^n \times \mathbb{R}^n$ of (4.21) with $i = 1, 2$, such that

$$x^1(\tau) = x^2(\tau), \quad p^1(\tau) \neq p^2(\tau). \quad (4.23)$$

According to [10] (see Ch.5, p.607-610) or [15], it follows that

Lemma 4.3. *Assume that the Hamiltonian system (4.21) is a complete system and also has no shock at all. If for every $\xi \in \mathbb{R}^n$, there exists a control $u : [t_0, T] \mapsto U$ such that the solution $(x(\cdot), p(\cdot))$ of (4.21) with the terminal condition*

$$x(T) = \xi, \quad (4.24)$$

satisfies

$$\dot{x}(t) = f(x(t), u(t), t), \quad \text{a.e. } t \in [t_0, T], \quad (4.25)$$

then $V : \mathbb{R}^n \times [t_0, T] \mapsto \mathbb{R}$ defined by

$$V(x(\tau), \tau) := \int_{\tau}^T F(x(s), u(s), s) ds, \quad (4.26)$$

is the value function of OCP.

Theorem 4.1. *Assume that (H5)-(H6) holds and the Hamiltonian system (4.21) is a complete system. If the control domain U is a compact, then **Hypotheses 3** holds.*

Proof By **Hypotheses 3**, there exists a unique control $u^* : [\tau, T] \mapsto U$ satisfying Pontryagin maximum principle under any initial data (x^0, τ) . Thus $\tilde{V} : \mathbb{R}^n \times [t_0, T] \mapsto \mathbb{R}$ is well-defined as follows:

$$\tilde{V}(x^0, \tau) := \int_{\tau}^T F(x^*(s), u^*(s), s) ds, \quad (4.27)$$

where $x^*(\cdot)$ is the solution of (1.2) with the initial condition $x(\tau) = x^0$ and the control $u^*(\cdot)$. Meanwhile, it follows from (4.16) and the assumptions that, the unique solution $(x^*(\cdot), p^*(\cdot))$ of the control Hamiltonian system (1.5)-(1.6) with $u(\cdot) = u^*(\cdot)$ is also the unique solution of the Hamiltonian system (4.21) with the terminal condition

$$x(T) = x^*(T).$$

The assumptions and Lemma 4.3 tell us that we only need to prove the Hamiltonian system (4.21) has no shock at all. Otherwise, if there are two solutions $(x^i(\cdot), p^i(\cdot))$ of the Hamiltonian system (4.21) with $i = 1, 2$, such that

$$x^1(\tau) = x^2(\tau), \quad p^1(\tau) \neq p^2(\tau), \quad (4.28)$$

for some $\tau \in [t_0, T]$, then it follows from (4.16) and the famous Filippov's Lemma in [18] (known as Measurable Selection Theorem) that there exist two admissible controls $u^i : [\tau, T] \mapsto U$ with $i = 1, 2$, (i.e., both $u^1(\cdot)$ and $u^2(\cdot)$ are Lebesgue measurable) such that both $u^1(\cdot)$ and $u^2(\cdot)$ satisfies the Pontryagin maximum condition together with $(x^1(\cdot), p^1(\cdot))$ and $(x^2(\cdot), p^2(\cdot))$, respectively, under the same initial data $(x^0, \tau) = (x^1(\tau), \tau)$. This is contradictory to **Hypotheses 3**. The proof is completed. \square

Remark 4.2. In this paper, we take the admissible control set at the initial time $\tau \in [t_0, T]$ as $\mathcal{U}(\tau)$ defined in (2.12), i.e., all Lebesgue measurable functions on $[\tau, T]$. For other types of admissible control set such as all piecewise continuous controls, the conclusions in this paper are also valid.

5 The conclusions

In this paper, we study in detail three hypotheses on the optimal control theory proposed by L. I. Rozonoer [1]. **Hypotheses 3** is only considered for the case with the smooth Hamiltonian. Now we are considering the case with the non-smooth Hamiltonian.

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